[1] Assume that a rigid body of mass m is in sliding motion maintaining contact with the inner wall surface of a hollow cylinder and a smooth frictionless horizontal surface without rolling, as shown in Fig. 1. The centroid (center of gravity) of the rigid body moves along a circle of radius R. The dimensions of the rigid body are sufficiently small compared to the radius of the sliding surface on the inner wall of the hollow cylinder, and the rotational moment of inertia of the rigid body is assumed to be negligible. Fill the blank spaces (*a*) through (*p*) in the text below regarding the motion of the rigid body with appropriate mathematical expressions. The constants m, R, μ, the variable $\theta(t)$ and its time derivatives $\dot{\theta}(t)$, $\ddot{\theta}(t)$ (representing $d\theta/dt$ and $d^2\theta/dt^2$, respectively) may be used in the answers as needed, after the definition of each is given in the text.

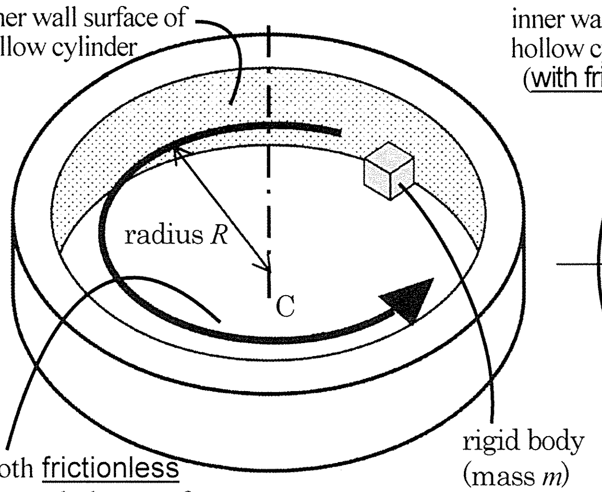

Fig. 1

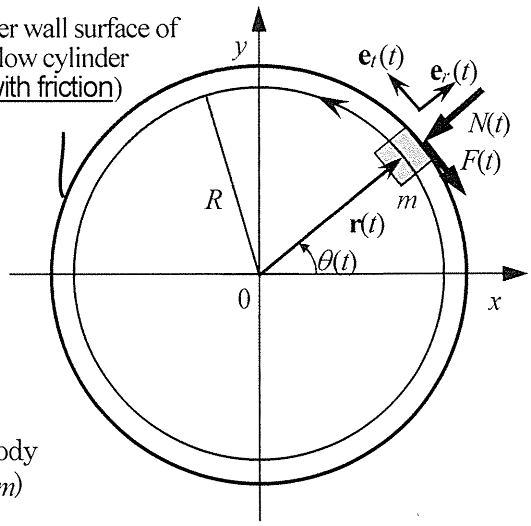

Fig. 2

[1-1] Let us find the normal force acting on the rigid body from the sliding surface on the hollow cylinder denoted by $N(t)$, and the friction force acting on the rigid body from the sliding surface on the hollow cylinder denoted by $F(t)$ when the rigid body is in counterclockwise motion along the sliding surface. Consider an xy Cartesian coordinate system on the horizontal plane with the origin at the location of the central axis of the sliding surface on the hollow cylinder (C), as shown in Fig. 2. The position of the rigid body at time t is represented by $\theta(t)$ defined as the

angle of the line connecting C to the centroid of the rigid body measured from the x-axis in the counterclockwise direction. If the rigid body continues counterclockwise circular motion after completing one round (or more rounds) along the sliding surface, the coordinate $\theta(t)$ shall increase as a continuous function of t and can take values greater than 2π.

Let us denote the two-dimensional position vector of the centroid of the rigid body by $\mathbf{r}(t)$ in the xy-coordinate system on the plane. The velocity vector and the acceleration vector are denoted by $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$, respectively, since they are first and second time derivatives of the position vector, respectively. Let $\mathbf{e}_r(t)$ and $\mathbf{e}_t(t)$ be the unit vectors with the same orientation as $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, respectively, which are perpendicular to each other and are expressed by

$$\mathbf{e}_r(t)=\begin{Bmatrix}\cos\theta(t)\\ \sin\theta(t)\end{Bmatrix}, \quad \mathbf{e}_t(t)=\begin{Bmatrix}\boxed{\quad (a) \quad}\end{Bmatrix}. \qquad (1)$$

The position vector $\mathbf{r}(t)$, velocity vector $\dot{\mathbf{r}}(t)$, and acceleration vector $\ddot{\mathbf{r}}(t)$ can be expressed by the following expressions using these unit vectors $\mathbf{e}_r(t)$ and $\mathbf{e}_t(t)$.

$$\mathbf{r}(t)=R\,\mathbf{e}_r(t) \qquad (2)$$

$$\dot{\mathbf{r}}(t)=\boxed{\quad (b) \quad}\,\mathbf{e}_t(t) \qquad (3)$$

$$\ddot{\mathbf{r}}(t)=\boxed{\quad (c) \quad}\,\mathbf{e}_r(t)+\boxed{\quad (d) \quad}\,\mathbf{e}_t(t) \qquad (4)$$

The direction of the normal force (magnitude $N(t)$) acting on the rigid body from the sliding surface in the hollow cylinder is opposite to that of the position vector, and the direction of the friction force (magnitude $F(t)$) acting on the rigid body from the sliding surface in the hollow cylinder is opposite to that of the velocity vector. Therefore, the equation of motion of the rigid body is expressed with $N(t)$, $F(t)$ and the unit vectors $\mathbf{e}_r(t)$ and $\mathbf{e}_t(t)$ as

$$m\ddot{\mathbf{r}}(t)=\boxed{\quad (e) \quad}. \qquad (5)$$

Substituting Eq.(4) into the left-hand side of Eq.(5), we have

$$\left(\boxed{\quad (f) \quad}\right)\mathbf{e}_r(t)+\left(\boxed{\quad (g) \quad}\right)\mathbf{e}_t(t)=0. \qquad (6)$$

The forces $N(t)$ and $F(t)$ can be derived from Eq. (6) considering that $\mathbf{e}_r(t)$ and $\mathbf{e}_t(t)$ are mutually perpendicular and linearly independent.

$$N(t) = \boxed{\quad (h) \quad} \tag{7}$$

$$F(t) = \boxed{\quad (i) \quad} \tag{8}$$

On the other hand, $F(t)$ and $N(t)$ are related by the following equation:

$$F(t) = \mu N(t) \tag{9}$$

where μ is the friction coefficient of the sliding surface on the hollow cylinder and the rigid body. Eliminating $N(t)$ and $F(t)$ from Eqs. (7), (8) and (9), we have the equation of motion of the rigid body in terms of the coordinate $\theta(t)$.

$$\ddot{\theta}(t) + \boxed{\quad (j) \quad} = 0 \tag{10}$$

[1-2] As the next step, assume that an initial velocity v_0 is given to the rigid body at the position $\theta = 0$ and time $t = 0$ in the direction along the sliding surface on the hollow cylinder, and the motion of the rigid body in the counterclockwise direction is initiated. Let us find the subsequent motion of the rigid body. The kinetic energy of the rigid body, denoted by T, can be expressed in terms of $\dot{\theta}$ as

$$T = \boxed{\quad (k) \quad} . \tag{11}$$

Using Eq. (11), $N(t)$ given by Eq. (7) can be expressed in terms of T without using $\dot{\theta}$. Substitution of the result into Eq. (9) yields the expression for $F(t)$ in terms of T.

$$F(t) = \boxed{\quad (l) \quad} \tag{12}$$

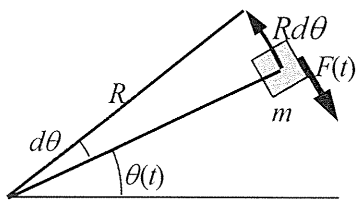

Fig. 3

On the other hand, loss of the kinetic energy of the rigid body during its motion of a small increment of position $d\theta$, as shown in Fig. 3, is equal to the work done by the frictional force $F(t)$ on the rigid body during the motion. Therefore, increment of the kinetic energy of the rigid body by the motion of $d\theta$ can be expressed by the following equation:

$$dT = -FRd\theta \tag{13}$$

in which the change of $F(t)$ during the motion of $d\theta$ is assumed to be negligible. Substituting Eq. (12) into Eq. (13) and taking the limit of $d\theta$ to zero, we obtain a first-order ordinary differential equation for T as a function of θ.

$$\frac{dT}{d\theta} = \boxed{\quad (m) \quad} \tag{14}$$

The solution of Eq. (14), $T(\theta)$, is obtained as an expression using v_0, considering the initial condition $T(0) = (1/2)\, mv_0^2$.

$$T(\theta) = \boxed{\quad (n) \quad} \tag{15}$$

Furthermore, substituting Eq. (11) into the left-hand side of Eq. (15) to eliminate T, we obtain the first-order ordinary differential equation for $\theta(t)$.

$$\dot{\theta} = \boxed{\quad (o) \quad} \tag{16}$$

The motion of the rigid body can be obtained by finding $\theta(t)$ as the solution of Eq. (16) that satisfies the initial condition $\theta(0) = 0$.

$$\theta(t) = \boxed{\quad (p) \quad} \tag{17}$$

From this result, the motion of the rigid body is found to be such that the rigid body does not cease to move for any passage of time t, although the velocity gradually decreases from the initial velocity v_0. It also implies that there is no upper limit on the number of rounds of the circular motion along the sliding surface on the hollow cylinder, as long as the relationship Eq. (9) is assumed.

[2] Fill in the blanks (A) to (P) below for the motion of a pendulum. Ignore the size of the mass, the mass of the string, the rotation around the string, the stretching and shrinking of the string, friction, and any other losses. It is assumed that the string does not bend. In the following, the coordinate system takes the *y*-axis vertically upward and the *x*-axis horizontally rightward, the MKS unit system and angular radians are used, and the gravity acceleration is denoted by g. The answers should be expressed only with the symbols shown in parentheses on the right side of the equation, although it is not necessary to use all symbols.

[2-1] As shown in Fig. 4, a point mass with mass m is suspended from point O by a string of length l. Consider the small oscillation of the point mass along arc s in the xy plane when the point mass lifted by hand is released smoothly while the string is straight. The angle of the string from the *y*-axis defines θ.

The forces acting on the point mass are the gravity F_G acting vertically downward and the tension F_T of the string, and the resultant force F_O acting in the tangential direction of arc s. The magnitude of F_O is expressed as follows.

$$F_O = \boxed{\text{(A)}} \quad (\theta, g, l, m) \qquad (1)$$

Considering the oscillation of the point mass along arc s, the equation of motion for θ can be derived

$$\frac{d^2\theta}{dt^2} = \boxed{\text{(B)}}. \quad (\theta, g, l, m) \qquad (2)$$

Assuming that the small infinitesimal oscillation $\theta \ll 1$, the position of the *x*-component can be approximated as $\boxed{\text{(C)}}$ (θ, g, l, m). From this, Eq. (2) can be rewritten as an expression for a simple harmonic oscillation as follows

$$\frac{d^2\theta}{dt^2} = \boxed{\text{(D)}}. \quad (\theta, g, l, m) \qquad (3)$$

Next, the solution of Eq. (3) can be expressed as $\theta = A\cos(\omega t + \delta)$ with constant A, angular frequency ω and constant δ. If the point mass at rest is released gently from the initial angle θ_0 at t =0, the angle θ and angular frequency ω at time t are given as follows.

$$\theta = \boxed{\text{(E)}} \quad (t, \theta_0, \omega) \qquad (4)$$

$$\omega = \boxed{\text{(F)}} \quad (\theta, g, l, m) \qquad (5)$$

Next, consider the case where the forcing term F is added to the right-hand side of Eq. (3),

$$F = B\cos\omega_F t, \qquad (6)$$

where B and ω_F are constants, and $\omega_F \neq \omega$. It is assumed that the motion of the point mass becomes the oscillation of the form

$$\theta = C\cos\omega_F t, \tag{7}$$

where C is a constant. The constant C is expressed by

$$C = \boxed{\text{(G)}} \quad (B, \omega, \omega_F) \tag{8}$$

Eq. (8) shows that as the angular frequency of the forcing term ω_F approaches to the natural frequency ω of the pendulum, the amplitude of the oscillation increases.

[2-2] As shown in Fig. 5, two-point masses with masses m_1 and m_2 are suspended from point O using strings of lengths l_1 and l_2 respectively. In this case, the motion of point mass 2 can be regarded as the pendulum motion seen from the position of point mass 1, and the motion of point mass 1 can be regarded as the pendulum motion seen from the point O considering the force acting from point mass 2. Let x_1 and x_2 be the displacements in the x-axis direction of each point mass, and θ_1 and θ_2 be the angles. Based on the Eq. (3), the equations of motion of the coupled single oscillation for the displacements x_1 and x_2 are expressed as follows.

$$m_1 \frac{d^2 x_1}{dt^2} = \boxed{\text{(H)}} - \boxed{\text{(I)}} \quad (x_1, x_2, g, l_1, l_2, m_1, m_2) \tag{9}$$

$$m_2 \frac{d^2 x_2}{dt^2} = \boxed{\text{(I)}} \quad (x_1, x_2, g, l_2, m_2) \tag{10}$$

Consider the case where two-point masses oscillate with the same angular frequency ω and phase δ. In this case, the displacements x_1 and x_2 can be expressed as follows.

$$x_1 = A\cos(\omega t + \delta) \tag{11}$$

$$x_2 = B\cos(\omega t + \delta) \tag{12}$$

where A and B are constants. Substituting Eqs. (11) and (12) into Eqs. (9) and (10), the following equations for A and B can be obtained.

$$\boxed{\text{(J)}}\, A + \boxed{\text{(K)}}\, B = 0 \quad (g, l_1, l_2, m_1, m_2, \omega) \tag{13}$$

$$\boxed{\text{(L)}}\, A + \boxed{\text{(M)}}\, B = 0 \quad (g, l_1, l_2, m_1, m_2, \omega) \tag{14}$$

Eliminating A and B, following relations for the angular frequency ω, masses m_1 and m_2, and string lengths l_1 and l_2 can be obtained.

$$\boxed{\text{(N)}} = 0 \quad (g, l_1, l_2, m_1, m_2, \omega) \tag{15}$$

There exist two positive values of ω that satisfy Eq. (15), ω_1 and ω_2. If the masses of the two-point masses are $m_1 \gg m_2$, then ω_1 and ω_2 can be obtained from Eq. (15).

$$(\omega_1, \omega_2) = \left(\sqrt{\frac{g}{l_1}}, \sqrt{\frac{g}{l_2}}\right) \tag{16}$$

From Eq. (13) or (14), the ratio of the constants A and B, which represents the magnitude of the oscillation of the two-point masses, can be obtained as follows.

$$\frac{A}{B} = \boxed{\text{(O)}}, \boxed{\text{(P)}} \quad (g, l_1, l_2) \tag{17}$$

Eq. (17) shows that for the oscillation with the larger absolute value of the ratio of constants A and B, the displacements of the two-point masses are either in the same direction or in the opposite direction, depending on how long or short the lengths of the two strings are.

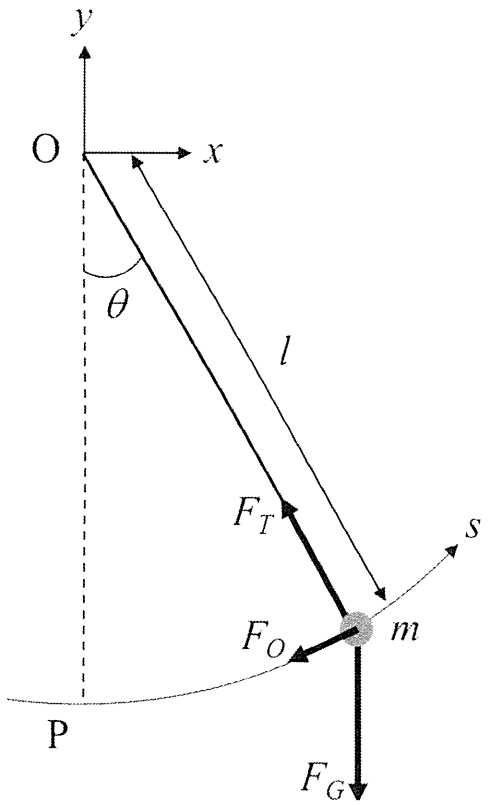

Fig. 4

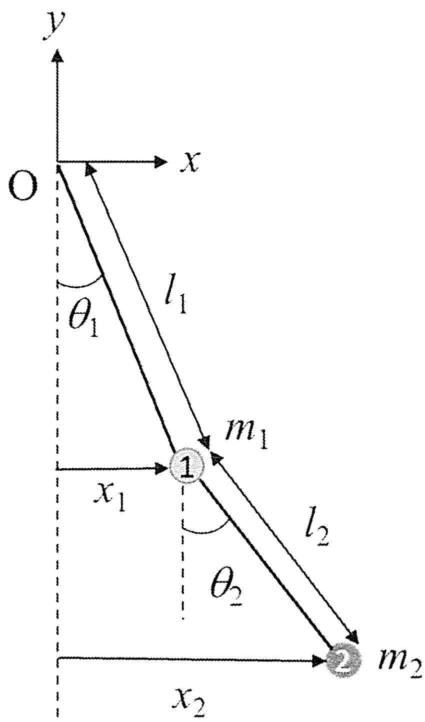

Fig. 5