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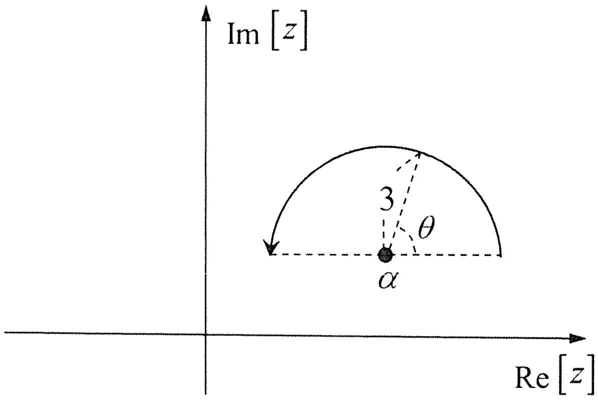
[1] Answer the following questions [1-1] and [1-2].

[1-1] Let us consider the complex integral $\int_C (z-\alpha)^2\,dz$. The integral route C is the semicircle with a radius of 3 centered at α as shown in the below figure. The integral direction is indicated by the arrow in the below figure. $\mathrm{Re}[z]$, $\mathrm{Im}[z]$ in the figure denote real and imaginary parts of z, respectively. θ is the argument of the complex number $(z-\alpha)$. Answer the following questions (1) and (2).

(1) Transform $\int_C (z-\alpha)^2\,dz$ to the definite integral with respect to θ.

(2) Calculate $\int_C (z-\alpha)^2\,dz$ using the result of (1).



[1-2] $f(t)$ is the piecewise smooth and absolutely integrable function defined for $-\infty < t < \infty$. The Fourier transform of $f(t)$ is defined as $F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-ikt}\,dt$. Introducing the piecewise smooth and absolutely integrable functions, $g(t)$ and $h(t)$, the convolution of them is defined as $(g*h)(t) = \int_{-\infty}^{\infty} g(s)h(t-s)\,ds$. Derive the equation $F\left[(g*h)(t)\right] = F\left[g(t)\right]F\left[h(t)\right]$.

[2] Let X denote the continuous random variable and $f_X(x)$ is the probability density function of the variable X. Answer the following questions.

[2-1] Suppose that the expectation and the variance of the random variable X denoted by $E(X)$ and $V(X)$ are given as $E(X) = \mu$ and $V(X) = \sigma^2$ respectively. Express the expectation and the variance of the random variable $Y = aX + b$, where a and b are the real constants.

[2-2] Let us introduce the function of the random variable $g(X) = e^{hX}$ and the expectation $M(h)$ is given by Equation (1).

$$M(h) = E(e^{hX}) = \int_{-\infty}^{\infty} e^{hx} f_X(x)dx \qquad (1)$$

in which $M(h)$ is the function of h. When the function $M(h)$ defined by Equation (1) exists, the function is denoted as the moment generating function. The moment generating function can identify the probability distribution of the random variable and can generate the moment of any order. When the random variable X follows the normal distribution $N(\mu, \sigma^2)$, the corresponding moment generating function $M(h)$ is given by Equation (2).

$$M(h) = E(e^{hX}) = e^{\mu h + \sigma^2 h^2/2}. \qquad (2)$$

Prove that the random variable $Y = 2X + 3$ follows the normal distribution using the above.

[2-3] Assuming that the random variable X follows the normal distribution $N(\mu, \sigma^2)$, prove Equation (2). The probability density function of the normal distribution $N(\mu, \sigma^2)$ is given by Equation (3).

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad (-\infty < x < \infty). \qquad (3)$$

[3] Answer the questions [3-1] through [3-3] for the following initial and boundary value problem.

Partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (0 < x < 2, 0 < t < \infty)$$

Boundary condition:

$$u(0,t) = u(2,t) = 0 \qquad (0 < t < \infty)$$

Initial condition:

$$u(x,0) = u_0 \qquad (0 \le x \le 2)$$

in which u_0 is constant.

[3-1] Using the separation of variables method, derive two ordinary differential equations for x and t from the given partial differential equation.

[3-2] Solve the ordinary differential equations derived in [3-1], and find the solutions of the partial differential equation satisfying the boundary condition. The solution can contain an integral constant.

[3-3] By determining the integral constant of the solution obtained in [3-2] to satisfy the initial condition, derive the following solution of the partial differential equation.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}(1 - \cos n\pi)\exp\left(-\frac{n^2\pi^2}{4}t\right)\sin\frac{n\pi}{2}x$$

where n is an integer. The following equation can be used for derivation.

$$\int_0^2 \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}dx = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases}$$

in which m and n are integers.