[1] As shown in Figure 1-1, a ball of mass m is projected from a horizontal surface with an initial speed v_0 at an angle θ $\left(0 < \theta < \frac{\pi}{2}\right)$ to the horizontal direction. Let the x-axis be horizontal and the y-axis be vertical and positive upward. The initial position of the ball is taken as the origin. Let the gravitational acceleration be g. The ball moves as in Figure 1-1, subjected to a downward gravitational force mg and an air resistance proportional to its speed, mkv, where v is the speed, k is the proportional constant per unit mass. Consider the velocity and position of the ball at time t.

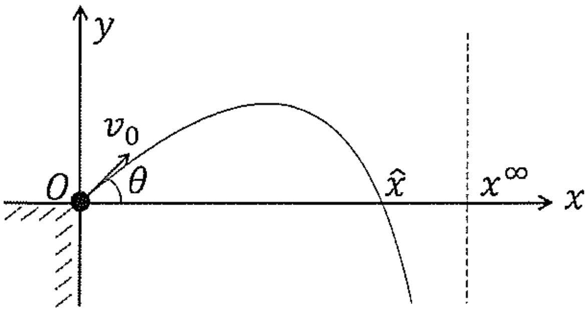

Figure 1-1

Fill in the blanks from $\boxed{\text{(A)}}$ to $\boxed{\text{(O)}}$ with the appropriate mathematical expressions or values. Note that ┆ ┆ represents the same item as $\boxed{}$. Let v_x and v_y be x- and y-components of the velocity, respectively.

The equation of motion in the x-direction is expressed using v_x as follows.

$$m\frac{dv_x}{dt} = \boxed{\qquad\text{(A)}\qquad} \tag{1-1}$$

Likewise, the equation of motion in the y-direction is expressed using v_y as follows.

$$m\frac{dv_y}{dt} = \boxed{\qquad\text{(B)}\qquad} \tag{1-2}$$

Solving the differential equation (1-1) and applying the initial condition, the x-component of velocity at time t is expressed as follows.

$$v_x = \boxed{\quad\text{(C)}\quad} \tag{1-3}$$

Likewise, solving the differential equation (1-2), the y-component of velocity at time t is expressed as follows.

$$v_y = \boxed{\quad\text{(D)}\quad} \tag{1-4}$$

Integrating Equations (1-3) and (1-4) with respect to time t, the positions x and y at time t are obtained as follows.

$$x = \boxed{\quad\text{(E)}\quad} \tag{1-5}$$

$$y = \boxed{\quad\text{(F)}\quad} \tag{1-6}$$

After a sufficiently long time, from Equations (1-3) and (1-4), the velocity of the ball approaches the terminal velocity $\boxed{\text{(G)}}$ and from Equation (1-5), the horizontal position approaches to $x^{\infty} = \boxed{\quad\text{(H)}\quad}$, beyond which the ball does not go any further.

Next, consider the horizontal distance when the ball passes through the same height as the point of projection, $\hat{x}$. Eliminating t from Equations (1-5) and (1-6), the trajectory of the ball is obtained as follows.

$$y = \left(\boxed{\quad\text{(I)}\quad}\right)x + \frac{g}{k^2}\ln\left(\boxed{\quad\text{(J)}\quad}\right) \tag{1-7}$$

The horizontal distance $\hat{x}$ is obtained by solving $y = 0$ in Equation (1-7) for x.

Consider the angle of projection that maximizes the horizontal distance $\hat{x}$ when the proportional constant of the air resistance k is sufficiently small. Approximating Equation (1-7) using the Taylor expansion of the logarithmic function $\ln(1 - z) = -z - z^2/2 - z^3/3 - \cdots$ up to the second order, the trajectory of the ball follows a parabolic function $y = \boxed{\text{(K)}}\, x - \boxed{\text{(L)}}\, x^2$. Therefore, the horizontal distance $\hat{x} = \boxed{\quad\text{(M)}\quad}$ is maximized when the angle of projection is $\theta = \boxed{\text{(N)}}$, being the same result as that in the case of no air resistance.

Consider the Taylor expansion of the logarithmic function in Equation (1-7) up to the third order. The condition $\frac{d\hat{x}}{d\theta} = \boxed{\text{(O)}}$ must be satisfied when the horizontal distance $\hat{x}$ reaches a maximum with respect to the angle of projection θ. However, this condition does not hold when $\theta = \boxed{\text{(N)}}$, indicating that this angle of projection cannot maximize the horizontal distance in this case.

[2] Answer the following questions regarding the motion of rigid rods in the xy-plane. The effect of gravity is assumed to be negligible. A single dot above a symbol denotes a first-order time derivative, and a double dot denotes a second-order time derivative.

[2-1] Fill in the blanks from $\boxed{\text{(a)}}$ to $\boxed{\text{(k)}}$ with appropriate mathematical expressions.

As shown in Figure 2-1, a rigid rod OA of length L is placed along the x-axis. The rod is hinged at the endpoint O and can rotate smoothly. A spring with spring constant k is attached to the other endpoint A.

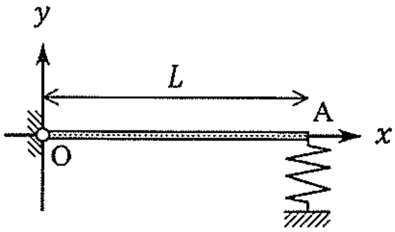

Figure 2-1

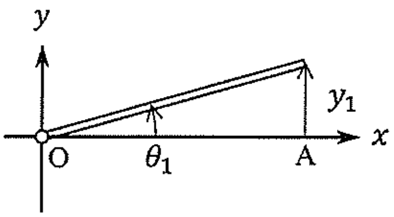

Figure 2-2

As shown in Figure 2-2, let the displacement in the y-direction at point A be y_1, and the rotation angle of the rod be θ_1. The rotation angle is set positive counterclockwise. Assuming that the displacement y_1 is sufficiently small compared to the length L of the rod, the rotation angle θ_1 can be approximated in terms of y_1 as $\boxed{\text{(a)}}$. Given that the restoring force of the spring is ky_1, the equation of rotational motion of the rod about point O can be written as follows in terms of the moment of inertia I_O about point O.

$$I_O\ddot{\theta}_1 = \boxed{\text{(b)}}$$

The rod undergoes periodic motion. The angular frequency of the motion is $\omega = \boxed{\text{(c)}}$.

As shown in Figure 2-3, a rigid rod AB is placed along the extension of the rigid rod OA from Figure 2-1. The rigid rods OA and AB have the same dimensions and mass density distribution. The two rods are connected by a hinge at point A and can rotate smoothly. A spring with spring constant k is attached to the endpoint B.

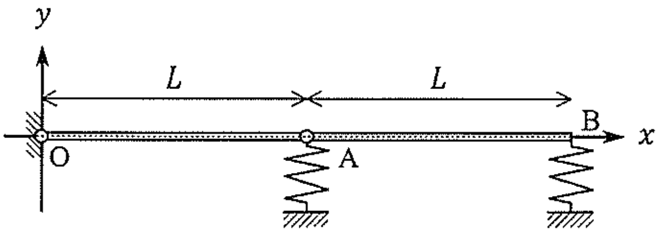

Figure 2-3

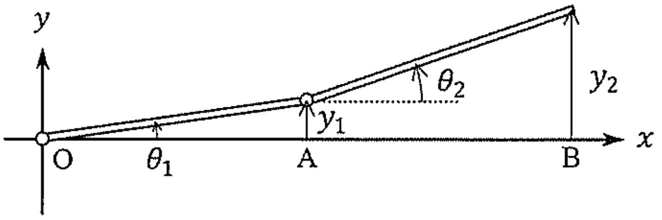

Figure 2-4

As shown in Figure 2-4, let the displacement in the y-direction at point B be y_2, and the rotation angle of the rigid rod AB be θ_2. Assuming that both displacements y_1, y_2 are sufficiently small compared to the length L of the rods, the rotation angle θ_2 can be approximated by $\boxed{\quad (\mathrm{d}) \quad}$.

The restoring force ky_1 acting at point A can be divided into two forces: F_1 and F_2, acting on the rigid rod OA and AB, respectively. These forces satisfy $F_1 + F_2 = ky_1$. The restoring force ky_2 acts at point B.

Given the applied forces, the equation of rotational motion of the rigid rod OA about point O can be written as follows.

$$I_O\ddot{\theta}_1 = \boxed{\quad (\mathrm{e}) \quad} \tag{2-1}$$

When the center of mass of the rigid rod AB is at the midpoint of the rod, the equation of rotational motion about the center of mass can be written in terms of the moment of inertia I_C of the rod about its center of mass as follows.

$$I_C\ddot{\theta}_2 = \boxed{\quad (\mathrm{f}) \quad} \tag{2-2}$$

The displacement of the center of mass of the rigid rod AB in the y-direction can be written as

$(y_1 + y_2)/2$, and the equation of motion of the center of mass can be written as follows in terms of the mass M of the rod.

$$M\frac{\ddot{y}_1 + \ddot{y}_2}{2} = \boxed{\quad (\mathrm{g}) \quad} \tag{2-3}$$

By eliminating F_1 and F_2 from Equations (2-1), (2-2), and (2-3), the following two equations are obtained.

$$\boxed{\quad (\mathrm{h}) \quad}\ \ddot{y}_1 + \boxed{\quad (\mathrm{i}) \quad}\ \ddot{y}_2 = -ky_1 \tag{2-4}$$

$$\boxed{\quad (\mathrm{j}) \quad}\ \ddot{y}_1 + \boxed{\quad (\mathrm{k}) \quad}\ \ddot{y}_2 = -ky_2 \tag{2-5}$$

These two equations indicate that the two rods undergo coupled oscillation.

[2-2] Write the equation relating the two moments of inertia, I_{O} and I_{C}.

[2-3] Equations (2-4) and (2-5) can also be derived using Lagrange's equations. Write the difference, $L = K - U$, between the kinetic energy K and the potential energy U, which is required for this derivation. Note that θ_1 and θ_2 should not be included.

[2-4] From Equations (2-4) and (2-5), the two natural angular frequencies of the coupled oscillation and their corresponding natural modes of oscillation can be obtained. Describe the procedure for obtaining natural angular frequencies and natural modes. It is not necessary to find the specific values. Note that Equations (2-4) and (2-5) can be written as $M\boldsymbol{A}\ddot{\boldsymbol{y}} = -k\boldsymbol{y}$ using the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{y}$. The vector $\boldsymbol{y}$ is given as follows.

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$