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[1] The curve C is defined by the following equation using orthogonal coordinates (x, y).

$$C(x, y) = (x^2 + y^2)^2 - (x^2 - y^2) = 0$$

[1-1] Express the curve C in polar coordinates (r, θ).

[1-2] Find the maximum value of x on the curve C.

[1-3] Find the maximum value of y on the curve C.

[1-4] Draw the curve C on the xy-plane.

[1-5] Find the area S surrounded by the curve C.

[2] The Laplace transform of $x(t)$ is defined by the following equation. Note that $s > 0$.

$$X(s) = \mathcal{L}\{x\}(s) = \int_0^{\infty} x(t)e^{-st}\,dt$$

[2-1] Find the Laplace transform of the following function.

$$x(t) = \mathrm{e}^{-\alpha t} \quad (\alpha \geq 0)$$

[2-2] Find the Laplace transform of the following function.

$$x(t) = t$$

[2-3] Find the Laplace transform of the following differential equation with the conditions, $x = 0$ and $\frac{dx}{dt} = -1$, when $t = 0$.

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 2e^{-3t} \qquad (1)$$

[2-4] Solve the equation (1) with the conditions, $x = 0$ and $\frac{dx}{dt} = -1$, when $t = 0$.

[2-5] The Laplace transform of the convolution integral of $x(t)$ and $y(t)$ can be expressed by the following equation.

$$\mathcal{L}\left\{\int_0^{\infty} x(t-u)y(u)\,du\right\}(s) = X(s)Y(s)$$

where, $Y(s)$ is the Laplace transform of $y(t)$. Solve the following integral equation.

$$x(t) = 1 + \int_0^{\infty} e^{-(t-u)}x(u)\,du$$

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[3] The norm of real vector $\vec{x} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ is defined by the following equation.

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x^2 + y^2 + z^2}$$

[3-1] Let $\boldsymbol{Q}$ be a 3×3 orthogonal matrix (i.e. $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$), where T represents the transpose of matrix. Prove that $|\vec{x}| = |\boldsymbol{Q}\vec{x}|$.

[3-2] A symmetric matrix $\boldsymbol{A}$ can be expressed by the following equation using an orthogonal matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$.

$$\boldsymbol{A} = \boldsymbol{P}^T \boldsymbol{D} \boldsymbol{P}$$

Find $\boldsymbol{D}$, when $\boldsymbol{A}$ is given by the following equation.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad (2)$$

[3-3] Find the maximum value of $\dfrac{\vec{x} \cdot (\boldsymbol{A}\vec{x})}{\vec{x} \cdot \vec{x}}$ when $\vec{x} \neq \vec{0}$, and $\boldsymbol{A}$ is defined in equation (2).